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An Exact Result in Two-Dimensional Quantum Electrodynamics with Nonzero Fermion Mass (*).

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Summary. — In two-dimensional spinor electrodynamics with nonvanishing fermion mass it is proved—using tesults of Brandt—that $[j_0^{un}(x, t), j_1^{un}(y, t)] = +(i/\pi)e_0^2\partial_1\delta(x-y)$, where $j_\mu^{un}(k)$ is the exact unrenormalized electromagnetic current. Implications for the exact solution of the vacuum polarization are discussed.

1. – Introduction.

Since THIRRING (1), in 1957, found an exact solution for the Fermi interaction of a zero-mass Dirac field in two-dimensional space time, many people (2) have been interested in this field-theoretical model. Also two-dimensional electrodynamics with vanishing fermion mass, first considered by BIALYNICKI-BIRULA (3) and GLASER and JAKŠIĆ (4), and solved explicitly in terms of Green's functions by SCHWINGER, BROWN, THIRRING and WESS (5), attracted many

(*) Partly based on a seminar talk given at the University of Karlsruhe in November 1968.

(1) W. THIRRING: *Ann. of Phys.*, **3**, 91 (1958).

(2) See: A. WIGHTMAN: in *Cargèse Lectures in Theoretical Physics 1964/II*, edited by M. LEVY (New York, 1967).

(3) I. BIALYNICKI-BIRULA: *Nuovo Cimento*, **10**, 1150 (1958).

(4) V. GLASER and B. JACKŠIĆ: *Nuovo Cimento*, **11**, 877 (1959).

(5) J. SCHWINGER: *Lectures, Seminar Trieste 1962* (IAEA, 1963); *Phys. Rev.*, **128**, 2425 (1962); L. S. BROWN: *Nuovo Cimento*, **29**, 617 (1963); C. SOMMERFIELD: *Ann. of Phys.*, **26**, 1 (1963); W. THIRRING and J. WESS: *Ann. Phys.*, **27**, 331 (1964); Further

field theorists. A large number of solutions was found, which led to quite a a confused situation. This only reflected the fact that, in spite of the trivial physical content of these models, a consistent formulation of these models meets all the big difficulties of a four-dimensional relativistic field theory. Therefore it soon became clear that these models are excellent laboratories to study questions such as the definition of the current, gauge invariance, field operators at the same space-time point, etc., and existence of a unique vacuum state (Goldstone theorem). One can also test the assumptions of current algebras (6) and related theories (7).

In spite of all these very important successes, these models have the drawback of giving rise to a trivial S-matrix. To understand the complicated dynamical mechanism of particle physics, we need exact solutions of nontrivial field theories.

It is therefore worth-while to study nontrivial extensions of the abovementioned models. The most obvious generalization, in our case, is to give a nonzero mass to the fermion. The Federbush (2) model describing the theory of the Fermi interaction $\lambda j_{1\mu}\varepsilon^{\mu\nu}j_{2\nu}$ between two spinors of masses m_1 and m_2, respectively, is such an important generalization. It can be solved exactly and gives an S-matrix different from one.

The more difficult problem—the Thirring model with nonzero fermion mass—was first successfully attacked by BEREZIN (8). He was able to derive an exact solution in the so-called pseudoparticle space, but transformation to the physical-particle space remained a not properly solved problem. Other approaches (9,10) did not get further than deriving perturbation results.

Because of these enormous difficulties, GLIMM, HEPP, JAFFE and WIGHT-

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literature up to 1969: C. S. LAM: *Nuovo Cimento*, **34**, 637 (1964); H. FRIED: *Brown University Conference* (1965); J. TARSKI: *Journ. Math. Phys.*, 5, 1713 (1964); H. FRIED: *Nucl. Phys.*, **75**, 691 (1966); J. TARSKI and D. DUBIN: *Ann. of Phys.*, **43**, 263 (1967); G. VELO: *Nuovo Cimento*, 52 A, 1028 (1967); L. ZASTAVENKO: Dubna preprint P2-3113 (1967); M. K. VOLKOV: Dubna preprint E2-3266, P2-3270 (1967); F. SCHWABL, W. THIRRING and J. WESS: *Ann. of Phys.*, **44**, 200 (1967); H. RECHENBERG: Dissertation (1968); preprint (1969).

(6) C. R. HAGEN: *Nuovo Cimento*, 51 B, 169 (1967); 51 A, 1033 (1967); C. R. HAGEN and G. GURALNIK: N.Y.O.-2262, T.A.-171 preprint (1968); C. R. HAGEN: U.R.-875-246 preprint (1968).

(7) R. A. COLEMAN and J. W. MOFFAT: *Phys. Rev.*, **159**, 1306 (1967); C. G. CALLAN, R. F. DASHEN and D. H. SHARP: *Phys. Rev.*, **165**, 1883 (1968); S. COLEMAN, D. GROSS and R. JACKIW: Harvard preprint (1969).

(8) F. BEREZIN and V. SUSHKO: *Sov. Phys. JETP*, **21**, 5 (1965); *Žurn. Èksp. Teor. Fiz.*, **48**, 1293 (1965).

(9) P. L. F. HABERLER: *Acta Phys. Austriaca*, **25**, 350 (1967).

(10) I. BIALYNICKI-BIRULA: *Proceedings of Seminar on Unified Theories of Elementary Particles*, URPA-11 (1963); F. VERBEURE: *Nuovo Cimento*, 42 A, 269 (1966).

MAN [11], and others, have started to attack the problem *ab initio* by defining the field theory properly, *i.e.* by introducing a box cut- off and an ultra-violet cut-off. They were able to prove important things such as the self-adjointness of the Hamilton operator for the φ^4 theory and the Yukawa theory in two-dimensional space time [11] (some important results were also obtained in three-dimensional space time) and, more recently, the existence of Heisenberg picture fields and their vacuum expectation values [12] (the latter ones for a general class of cut-off model fields). The most recent paper even presents an existence proof for the φ^4 theory in two dimensions without cut-offs [13].

In this paper we would like to attack the problem from a somewhat different angle. Following the interesting works of ZIMMERMANN [14], WILSON [15] and BRANDT [16], we construct the exact current operator in quantum electrodynamics, taking carefully into account all the subtleties connected with two operators at the same space-time point, gauge invariance and the equal-time limit.

In Sect. 2 we review the main assumptions and the results of WILSON [15] and BRANDT [16,17]. In Sect. 3, we derive the exact expression for the current operator and we calculate the equal-time limits and discuss briefly the implications of our main result:

$$[j_1^{\text{un}}(x,t), j_1^{\text{un}}(y,t)] = +\frac{i}{\pi}e_0^2\partial_1\delta(x-y)\,, \tag{1.1}$$

where the index « un » means unrenormalized, and e_0 is the unrenormalized charge.

We give, for illustration, some applications in the Appendices.

2. – Review [18] of the work of Wilson and Brandt.

Wilson's [15,16] proposal amounts to the following. Any product $\chi(x)\cdot\chi_1(x_1)\dots\chi_n(x_n)$ of local field operators has, for all x_n near x, an expansion of

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([11]) For an extensive review, see A. WIGHTMAN: *Proceedings at the 1968 Rochester Conference in Vienna*; K. HEPP: Kiev preprint (1968); J. L. CHALLIFOUR: *Journ. Math. Phys.*, **9**, 1137 (1968); K. HEPP: Bures preprint (1968); I thank Prof. K. HEPP for sending me his work prior to publication.

([12]) A. JAFFE, O. E. LANFORD III and A. S. WIGHTMAN: E.T.H. preprint (1968).

([13]) A. JAFFE and J. GLIMM: Courant Institute preprints (1968, 1969); I thank Prof. A. JAFFE and Prof. J. GLIMM for sending me their preprints.

([14]) W. ZIMMERMANN: *Nuovo Cimento*, **10**, 597 (1958).

([15]) K. WILSON: unpublished Cornell Report; to be published in *Phys. Rev.*; I thank Prof. WILSON for sending me his work.

([16]) R. BRANDT: *Ann. of Phys.*, **44**, 221 (1967), and UMD 646.

([17]) R. BRANDT: *Ann. of Phys.*, 52, 122 (1969), and UMD 673; *Phys. Rev.*, **166**, 1795 (1968).

([18]) For clarity we present a quite detailed review. See also: R. BRANDT: UMD 910.

the form

$$\chi(x) \dots \chi_n(x_n) = \sum_{j=1}^{N} E_j(x, x_1, \dots, x_n)\, \chi'_j(x) \tag{2.1}$$

plus terms which vanish as $x_i \to x$, and the expansion is valid in the weak sense: one must sandwich the product $\chi(x) \dots \chi(x_n)$ between fixed final and initial states. The χ'_j's are also local operators. The complete expansion in general involves an infinite number of local fields χ'_j but to any finite order in $x - x_i$ only a finite number of fields contribute. The dimension of χ'_j is less than the dimension of $\chi(x), \dots, \chi_n(x_n)$ and the E_j's (which are matrices in the internal variables) are distributions in $x - x_i$, with dimensions

$$d_j = \left[\dim\left(\chi(x) \dots \chi_n(x_n)\right) - \dim \chi'_j\right] \tag{2.2}$$

and singularities

$$E_j = (x - x_i)^{-d_j} . \tag{2.3}$$

This is the most important assumption and it is taken from the work of KASTRUP and MACK [19], who assert that scale invariance is the most crucial broken symmetry. The importance of scale invariance for the analysis of short-distance behaviour is apparent in the power counting arguments of DYSON, and in the relation between the renormalizability of an interaction and its dimension [15].

Starting from these proposals, BRANDT [16,17] was able to derive a renormalized relativistic perturbation theory from finite local field equations for the neutral pseudoscalar meson theory and for spinor electrodynamics. For the latter, we present in the following a brief review where the main results of BRANDT [17] are translated into two-dimensional space-time [20].

The local formulation of spinor electrodynamics is based on the field equations [21]:

$$\left(i(\gamma\cdot\partial) - m\right)\psi(x) = f(x) = \lim_{\eta\to 0} f(x, \eta) , \tag{2.4}$$

$$\Box\, A_\mu(x) = j_\mu(x) = \lim_{\xi\to 0} j_\mu(x; \xi) , \tag{2.5}$$

with the subsidiary condition

$$\partial^\mu A_\mu^{(-)} |\varphi\rangle = 0$$

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[19] G. MACK: *Nucl. Phys.*, B 5, 499 (1968), and references cited therein.

[20] P. L. F. HABERLER: unpublished results.

[21] We use the metric $p^2 = p_0^2 - p_1^2$ with $g_{00} = -g_{11} = 1$. For further definitions, see ref. [9].

for physical states $|\varphi\rangle$. We shall therefore work within the usual Lorentz gauge Gupta-Bleuler [22] framework, but we shall not use the indefinite metric explicitly.

The currents are defined in the following way;

$$(2.6)\qquad j_\mu(x;\xi) = \exp\left[T\bar\psi(x)\gamma_\mu\psi(x+\xi) - C_{1\mu}(\xi) - C_{2\mu\nu}(\xi)A^\nu(x) - \right.$$
$$\left. - C_{3\mu\nu k}(\xi)A^{\nu,k}(x) - C_{4\mu\nu k\lambda}(\xi)A^{\nu,k\lambda}(x) - C_{6ij\mu}(\xi):\bar\psi_i(x)\psi_j(x):\right].$$

(The terms $C_{5\mu\nu k\lambda}(\xi)a^{\nu k\lambda}(x)$, corresponding to the four-photon coupling, and $C_{7\mu\nu k}:A^\nu A^k:$, $C_{8\mu\nu k\lambda}\partial^\lambda:A^\nu A^k:$ corresponding to diagrams with three external photon lines, which can be found in Brandt's paper [17], are left out, in anticipation of the results of Sect. 3).

$$(2.7)\qquad f(x,\eta) = \exp\left[T(\gamma\cdot A)(x+\eta)\psi(x) - D_1(\eta)\psi(x) - D_{2\mu}(\eta)\partial^\mu\psi(x) - D_3(\eta)f(x)\right]$$

where T means the time-ordered product.

For our purpose it is enough to study $j_\mu(x;\xi)$, since it contains all the necessary information to derive eq. (1.1) Some remarks concerning $f(x,\eta)$ will be made in Appendix A.

All the parameters and field operators are understood to be *renormalized.* The functions $D_i(\eta)$ and $C_i(\xi)$, which we refer to commonly as $E_i(\xi)$, have singularities for $\xi\to 0$, which compensate those of the local products, as $T\bar\psi(x)\gamma_\mu\psi(x+\xi)$ for example, so that the limit $\xi\to 0$ exists in (2.6). The operators occurring in (2.6) and (2.7) are all those in the theory with dimension (in mass units) 1 and $\frac12$, respectively.

We have

$$(2.8)\qquad \begin{cases} \dim\psi = \frac12\,, & \dim e = 1\,,\\ \dim A_\mu = 0\,, & \dim j_\mu(x) = \dim e + \dim\bar\psi\gamma_\mu\psi = 2\,,\\ \dim\partial_\mu = 1\,, & \dim\delta(x-x') = 1\,,\qquad \text{etc.}\end{cases}$$

From (2.2) and (2.8), it therefore follows that

$$(2.9)\qquad \begin{cases} \dim C_1^\mu(\xi) = 1\,, & \dim C_4^{\mu\nu k\lambda}(\xi) = -1\,,\\ \dim C_2^{\mu\nu}(\xi) = 1\,, & \dim C_6^\mu(\xi) = 0\,.\\ \dim C_3^{\mu\nu k}(\xi) = 0\,, & \end{cases}$$

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[22] S. N. Gupta: *Proc. Phys. Soc.*, **63**, 681 (1950); K. Bleuler: *Helv. Phys. Acta*, **23**, 567 (1950).

Since the leading singularities in perturbation theory are mass independent (following WILSON and BRANDT), we find for $E_i(\xi)$ ([23]) from (2.3)

$$(2.10)\qquad \begin{cases} C_1^{\mu} \sim \xi^{-1}\,, & C_4^{\mu\nu k\lambda} \sim \xi\,, \\ C_2^{\mu\nu} \sim \xi^{-1}\,, & C_{6ij}^{\mu} \sim 1\,, \\ C_3^{\mu\nu k} \sim 1\,, & \end{cases}$$

and

$$(2.11)\qquad \bar{\psi}(x)\gamma_\mu \psi(x+\xi) \sim \xi^{-1}$$

as $\xi \to 0$, within logarithmic factors. That this is true in our case (it is not an assumption), follows from the exact solution of the Thirring model for vanishing fermion mass ([24]). The renormalized quantity $\bar{\psi}(x)\gamma_\mu\psi(x+\xi)$ has exactly this behaviour. The generalized Wick products $:\bar{\psi}(x)\psi(x):$ must be defined by similar expansions. Possible arbitrariness in defining such products is just the usual arbitrariness of choosing basic vectors in a vector space and corresponds to the usual renormalization invariance.

The functions $E_i(\xi)$ can be essentially uniquely determined by imposing the usual renormalization conditions on the «primitively divergent» proper part functions:

$$(2.12)\qquad \Pi_{\mu\nu}(0) = \Pi'_{\mu\nu}(0) = \Pi''_{\mu\nu}(0) = 0\,,$$

$$(2.13)\qquad \Sigma(\gamma\cdot p = m) = \Sigma'(\gamma\cdot p = m) = 0\,,$$

$$(2.14)\qquad \Gamma_\mu(p, p')|_{\gamma\cdot p = \gamma\cdot p' = m,\ p = p'} = e\gamma_\mu\,,$$

$$(2.15)\qquad \Pi_{\alpha\beta\gamma\delta}(0, 0, 0, 0) = 0\,.$$

Here $\Pi_{\mu\nu}$ and Σ are the proper self-energy parts defined in terms of the photon and electron Green functions $D_{\mu\nu}$ and G by

$$(2.16)\qquad D_{\mu\nu}(k) = D_{F\mu\nu}(k) + D_{F\mu\varrho}\Pi^{\varrho\lambda}D_{\lambda\nu}(k)\,,$$

$$(2.17)\qquad D_{F\mu\nu}(k) = -g_{\mu\nu}/(k^2 + i\varepsilon)\,,$$

$$(2.18)\qquad [\gamma\cdot p - m - \Sigma(p)]G(p) = 1\,.$$

([23]) We cannot put $C_4^{\mu\nu\alpha\beta}(\xi)$ equal to zero because it multiplies a function which behaves like ξ^{-1} for $\xi \to 0$. This is due to the fact that $\langle 0|A_\mu^{\text{free}}(x)A_\nu^{\text{free}}(y)|0\rangle = g_{\mu\nu}D(x-y) \sim g_{\mu\nu}\log(x-y)^2$ for $x \sim y$. The derivative of $D(x-y)$ goes like $(x-y)^{-1}$ roughly speaking.

([24]) B. KLAIBER: *Boulder Lectures, 1967* (New York, 1968), p. 141; This work contains a beautiful exposition and solution of all the consistency problems in the Thirring model.

Γ_μ is the proper vertex part and $\Pi_{\alpha\beta\gamma\delta}$ the proper photon-photon scattering amplitude.

Conditions are imposed on the integral equations relating all the proper functions of the theory. For example one finds [20]:

$$\Pi^{\mu\nu}(k) = -ie\int\frac{d^2p}{(2\pi)^2}\big[\operatorname{tr}\gamma^\mu G(p)\Gamma^\nu(p,k)G(p-K) + i\,C_2^{\mu\nu}(p) - \\ - i\,C_4^{\mu\nu\alpha\beta}(p)k_\alpha k_\beta + i\,C^\mu_{6ij}\Pi'^{\nu}_{ji}(k)\big]\,. \tag{2.19}$$

A calculation of $\Pi_{\mu\nu}(k)$ to lowest order, taking into account all extra terms, will be given in Appendix C, with

$$\begin{cases} C_2^{\mu\nu}(p) \;\; = +\,i\operatorname{tr}\gamma^\mu G(p)\Gamma^\nu(p,0)G(p)\,, \\ C_4^{\mu\nu\alpha\beta}(p) = -\dfrac{i}{2}\partial_k^\alpha\partial_k^\beta\operatorname{tr}\gamma^\mu G(p)\Gamma^\nu(p,k)G(p-k)|_{k=0}\,, \end{cases} \tag{2.20}$$

$$\begin{cases} C^\mu_{6ij}(p) \;\; = ie^{-1}\operatorname{tr}\gamma^\mu G(p)H_{ij}(p,q,0)G(p)|_{\gamma\cdot q=m}\,, \\ C_1^\mu(p) \;\; = i\operatorname{tr}\gamma^\mu G(p) = iJ^\mu(p)\,, \end{cases} \tag{2.21}$$

where $C_1^\mu(p)$ was determined from the conditions

$$\langle 0|A^\mu|0\rangle = 0\,, \qquad \langle 0|j^\mu(x)|0\rangle = 0 \tag{2.22}$$

anu

$$\langle 0|:\bar\psi_i(x)\psi_j(x):|0\rangle = 0\,. \tag{2.23}$$

$H_{ij}(k,p,p')$ is the proper electron-electron scattering amplitude and $\gamma^\mu_{ij}\Pi'^{\nu}_{ji} = \Pi^{\mu\nu}$.

Iteration of this infinite set of coupled integral equations yields perturbation expansions (in terms of the renormalized charge e) for all the Green functions of the theory. So the limits in (2.6) and (2.7) do exist [17] and yield the correct finite local current operator [17,20].

The advantage of the above formalism is that it enables a direct imposition of local gauge invariance. One can show that the requirement that the field equations are invariant under the local gauge transformations:

$$\psi(x)\to\psi(x)\exp[-ie\alpha(x)]\,, \qquad A_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x)\,, \tag{2.24}$$

is equivalent to the requirement that the theory satisfies all the generalized Ward identities and divergence conditions like

$$-e\partial^\mu G(p) = G(p)\Gamma^\mu(p,0)G(p)\,, \tag{2.25}$$

$$-eD_2^\mu(k) \;= i\gamma^\mu D_3(k)\,. \tag{2.26}$$

Further, one can show that the theory determined by the normalizations (2.11)-(2.14) satisfies these conditions. For later reference, we show here some of these conditions:

$$k_\mu \Pi^{\mu\nu}(k) = 0 \,, \tag{2.27}$$

$$k_{1\alpha}\Pi^{\alpha\beta\gamma\delta}(k_1 k_2 k_3 k_4) = \ldots = k_4 \Pi = 0 \,, \tag{2.28}$$

$$C_2^{\mu\nu}(p) = -ie\partial^\nu J^\mu(p) \,, \tag{2.29}$$

$$C_3^{\mu\nu\alpha}(p) = -\frac{e}{2}\partial^\alpha\partial^\nu J^\mu(p) + \tilde{C}_3^{\mu\nu\alpha}(p) \,, \tag{2.30}$$

$$C_4^{\mu\nu\alpha\beta}(p) = \frac{ie}{6}\partial^\alpha\partial^\beta\partial^\nu J^\mu(p) + \tilde{C}_4^{\mu\nu\alpha\beta}(p) \,, \tag{2.31}$$

where

$$k_\nu k_\alpha \tilde{C}_3^{\mu\nu\alpha}(p) = 0 \,, \qquad k_\nu k_\alpha k_\beta \tilde{C}_4^{\mu\nu\alpha\beta}(p) = 0 \,. \tag{2.32}$$

We will see later that $\tilde{C}_3^{\mu\nu\alpha}$ vanishes because of charge renormalization invariance. $\tilde{C}_4^{\mu\nu\alpha\beta}$ will be specified in the next Section.

One can then easily convince oneself [17] that (2.27)-(2.32) are necessary, as well as sufficient, in order that our functions satisfy the requirements of gauge invariance. In the following we will show that we can specify the current completely from the gauge invariance of the field equations, charge-conjugation invariance and the usual equal-time commutation relations, without even specifying integral equations, as we have indicated above.

3. – Determination of the current.

Let us now see what are the necessary and sufficient conditions which the currents $j_\mu(x)$ and $f(x)$ must satisfy so that the field equations (2.4) and (2.5) are invariant under the gauge transformation (2.24). It is easy to see that

$$j^\mu(x) \to j^\mu(x) \,, \tag{3.1}$$

$$f(x) \to \exp[-ie\alpha(x)]f(x) + e\big(\gamma^\mu \partial_\mu \alpha(x) \exp[-ie\alpha(x)]\psi(x)\big) \,. \tag{3.2}$$

Our task is now to derive the necessary and sufficient conditions which the functions $E_i(\xi)$ must satisfy so that (3.1) and (3.2) are satisfied.

If we apply (2.24) to (2.7), we just get

$$ieD_{2\mu}(\xi) = D_3(\xi)\gamma_\mu \,. \tag{3.3}$$

This is just the Fourier transform of (2.26), exactly what we wanted to get. In Appendix A, it is shown that this Ward identity is identically fulfilled in lowest order. [We now turn to the electric current operator $j_\mu(x)$, the main object of our interest.

We have to apply transformation (2.24) onto each term of eq. (2.6). The first term $\bar{\psi}(x)\gamma_\mu\psi(x+\xi)$ will require the factor

$$\exp[ie[\alpha(x)-\alpha(x+\xi)]] = \exp[-ie[\xi\alpha' + \tfrac{1}{2}\xi\xi\alpha'' + \tfrac{1}{6}\xi\xi\xi\alpha''' + \ldots]] = 1 - ie(\xi\alpha') + O(\xi^2)\,, \tag{3.4}$$

where $\xi\alpha' = \xi^\mu\partial_\mu\alpha(x)$, etc.

Notice that we truncated the expansion after the ξ term because we know from (2.11) that $\bar{\psi}(x)\gamma_\mu\psi(x+\xi)$ behaves like ξ^{-1} so that

$$|\xi|^2\bar{\psi}(x)\gamma_\mu\psi(x+\xi)\to 0\,.$$

We are now ready to investigate the restrictions on the C_i in (2.7) imposed by the requirement that $j_\mu(x)$ be gauge invariant. We shall first determine the the forms C_i must have, in order that $\langle 0|\delta j_\mu(x)|0\rangle = 0$ simply. As a basis for the usual induction argument, we can assume that

$$\langle 0|\delta:\bar{\psi}_i(x)\psi_j(x):|0\rangle = 0\,. \tag{3.5}$$

This is due to the fact that $j_\mu(x) = \gamma^\mu_{ij}:\bar{\psi}_j\psi_i:(x)$, so that invariance of $:\bar{\psi}_i\psi_j:(x)$ implies invariance of $j_\mu(x)$ and vice versa.

Using (3.4), (3.5) and (2.6), we get (taking literally eq. (8.39) of ref. [17], to demonstrate that C_5, C_7, C_8 really contribute to zero)

$$\begin{aligned}\langle 0|\delta j^\mu(x)|0\rangle = e\lim_{\xi\to 0}\Bigg\{\Bigg[-ie\left(\xi\alpha' + \frac{1}{2}\xi\xi\alpha'' + \frac{1}{6}\xi\xi\xi\alpha'''\right) - \\ -\frac{e^2}{2}(\xi\alpha'\xi\alpha' + \xi\alpha'\xi\xi\alpha'') + \frac{ie^3}{6}(\xi\alpha'\xi\alpha'\xi\alpha')\Bigg]iJ^\mu(\xi) - C_2^{\mu\nu}(\xi)\alpha_\nu - C_3^{\mu\nu k}(\xi)\alpha_{\nu k} - \\ - C_4^{\mu\ \lambda}(\xi)\alpha_{\nu k\lambda} - C_7^{\mu\nu k}(\xi)\alpha_\nu\alpha_k - C_8^{\mu\nu k\lambda}(\xi)\partial_\lambda\alpha_\nu\alpha_k - C_5^{\mu\nu k\lambda}(\xi)\alpha_\nu\alpha_k\alpha_\lambda\Bigg\}\,.\end{aligned} \tag{3.6}$$

Since (3.6) holds for arbitrary $\alpha(x)$, by requiring $\langle 0|\delta j^\mu(x)|0\rangle = 0$, we obtain the following conditions that the coefficients of the terms α_ν, $\alpha_{\nu k}$, $\alpha_{\nu k\lambda}$, $\alpha_\nu\alpha_k$, etc., vanish. Taking into account the total symmetry of $\alpha_{\nu k}$ and $\alpha_{\nu k\lambda}$ we get

$$C_2^{\mu\nu}(\xi) = e\xi^\nu J^\mu(\xi)\,, \tag{3.7}$$

$$C_3^{\mu\nu k}(\xi) = \frac{e}{2}\xi^\nu\xi^k J^\mu(\xi) + \tilde{C}_4^{\mu\nu k}(\xi)\,, \tag{3.8}$$

$$C_4^{\mu\nu k\lambda}(\xi) = \frac{e}{6}\xi^\nu\xi^k\xi^\lambda J^\mu(\xi) + \tilde{C}_4^{\mu\nu k\lambda}(\xi)\,, \tag{3.9}$$

where

$$x_\nu x_k \tilde{C}_3^{\mu\nu k}(\xi) = 0 , \qquad x_\nu x_k x_\lambda \tilde{C}_4^{\mu\nu k}(\xi) = 0 , \tag{3.10}$$

$$\left\{\begin{aligned} C_7^{\mu\nu k}(\xi) &= -\frac{ie^2}{2}\xi^\nu \xi^k J^\mu(\xi) , \\ C_8^{\mu\nu k\lambda}(\xi) &= -\frac{ie^2}{4}\xi^\nu \xi^k \xi^\lambda J^\mu(\xi) , \\ C_5^{\mu\nu k\lambda}(\xi) &= -\frac{ie^3}{6}\xi^\nu \xi^k \xi^\lambda J^\mu(\xi) . \end{aligned}\right. \tag{3.11}$$

We remember that

$$J^\mu(\xi) = -i\langle\bar{\psi}(x)\gamma^\mu\psi(x+\xi)\rangle = \operatorname{tr}\gamma^\mu G(\xi) . \tag{3.12}$$

Now from (2.10) and (2.11), we find [23]

$$C_3^{\mu\nu k}(\xi) = \tilde{C}_3^{\mu\nu k}(\xi) , \tag{3.13}$$

$$C_4^{\mu\nu k\lambda}(\xi) = \tilde{C}_4^{\mu\nu k\lambda}(\xi) , \tag{3.14}$$

$$C_5^{\mu\nu k\lambda}(\xi) = 0 , \tag{3.15}$$

$$C_7^{\mu\nu k}(\xi) = 0 , \tag{3.16}$$

$$C_8^{\mu\nu k\lambda}(\xi) = 0 , \tag{3.17}$$

where we have used the fact that $|\xi^2| J^\mu(\xi) = 0$. We get the important result that we do not need a renormalization of the photon-photon amplitude $\left(C_5^{\mu\nu k\lambda}(\xi) = 0\right)$. In Appendix B we show this explicitly by calculating the lowest-order contribution to this process. The fact that the three-photon vertex does not give any contribution was expected, as we know from Furry's theorem that these graphs vanish identically.

$\langle 0|\delta j^\mu(x)|0\rangle = 0$ implies (3.13)-(3.17). Conversely, if the C_i's have the form (3.13)-(3.17) then $\langle 0|\delta j^\mu(x)|0\rangle = 0$.

Let us now compare (3.13)-(3.17) with eqs. (2.27)-(2.32). We see that they are just the Fourier transforms of each other and, using the extensive arguments given by BRANDT, we see that eqs. (3.13)-(3.17) are the necessary conditions for $j_\mu(x)$ to be gauge invariant, *i.e.* $\langle 0|\delta j^\mu(x)|0\rangle = 0$ is equivalent to $\delta j^\mu(x) = 0$. This equivalence requires use of the fact that $\langle 0|j^\mu(x)|0\rangle = 0$, so that

$$C_1^\mu(\xi) = iJ^\mu(\xi) , \tag{3.18}$$

which is the Fourier transform of (2.21).

We can get another useful relation if we transform the fields using (3.2), (3.13)-(3.17) and $\delta:\bar{\psi}(x)\psi(x):=0$:

$$\bar{\psi}(x)\gamma^\mu\psi(x+\xi)\xi^\alpha \sim iJ^\mu(\xi)\xi^\alpha\,, \tag{3.19}$$

which we will need later.

We have now for $j_\mu(x;\xi)$ the following equation:

$$\begin{aligned} j_\mu(x;\xi) = e[T\bar{\psi}(x)\gamma_\mu\psi(x+\xi) - iJ_\mu(\xi)[1-ie\xi^\nu A_\nu(x)] - \\ - \tilde{C}_{3\mu\nu k}(\xi)A^{\nu_1 k}(x) - \tilde{C}_{4\mu\nu k\lambda}(\xi)A^{\nu_1 k\lambda}(x) - \operatorname{tr} C_6^\mu(\xi):\psi(x)\bar{\psi}(x):]\,. \end{aligned} \tag{3.20}$$

So $\tilde{C}_3^{\mu\nu k}(\xi)$, $\tilde{C}_4^{\mu\nu k\lambda}(\xi)$ and C^μ_{6ij} have still to be determined. Following BRANDT [17], we write [25]

$$\tilde{C}_4^{\mu\nu k\lambda}(\xi)A_{\nu_1 k\lambda}(x) = C(\xi^2)\partial_\nu F^{\mu\nu}(x)\,, \tag{3.21}$$

$$\tilde{C}_3^{\mu\nu k}(\xi)A_{\nu_1 k}(x) \;=\tilde{C}(\xi^2)\xi_\nu F^{\nu\mu}(x)\,. \tag{3.22}$$

To determine $C_6^\mu(\xi)$ further, let us use explicitly invariance under charge conjugation:

$$CA^\mu(x)C^{-1} = -A^\mu(x)\,, \tag{3.23}$$

$$Cj^\mu(x)C^{-1} \;= -j^\mu(x)\,. \tag{3.24}$$

From this we find, for $C_6^\mu(\xi)$

$$Cc_{6\mu}(\xi)C^{-1} = -C^T_{\mu 6}(-\xi)\,, \tag{3.25}$$

therefore

$$C_6^\mu(\xi) = K_1(\xi^2)\gamma^\mu + K^2(\xi^2)\xi^\mu\xi\cdot\gamma + K_3(\xi^2)\xi^\mu\,. \tag{3.26}$$

Now since $j_\mu(x)$ is a covariant quantity, and since we will impose analyticity properties of the perturbation theory on the C_i, *i.e.* excluding [18] terms like $\sqrt{\xi^2}$, we find from (3.23) and from the fact that C_6^μ diverges only logarithmically

$$C^\mu_{6ij}(\xi) = K_1(\xi^2)\gamma^\mu_{ij}\,, \tag{3.27}$$

---

[25] We would like to remark that eq. (3.20) is manifestly gauge invariant, if we use eqs. (3.22), (3.28) and $1-ie\xi\cdot A = :\exp\left[-ie\int_x^{x+\xi}\mathrm{d}\eta\cdot A(\eta)\right]:$ because one can truncate the expansion after the first two terms in our case. Therefore also the Wick ordering sign $::$ is trivial, contrary to the four-dimensional case.

and since

$$j^\mu(x) = e\gamma^\mu_{ij}:\bar\psi_i(x)\psi_j(x): , \qquad eC^\mu_{6ij}:\bar\psi_i(x)\psi_j(x): = K_1(\xi^2)\, j^\mu(x) . \tag{3.28}$$

One can now show ([17]), using (3.25), that

$$Cj^\mu(x;\xi)\,C^{-1} = -j^\mu(x+\xi_1-\xi) + \text{ terms which go to zero as } \xi \text{ goes to zero.} \tag{3.29}$$

With

$$j^\mu(x+\xi_1-\xi) = e\bar\psi(x+\xi)\gamma^\mu\psi(x) + eJ^\mu\big(i - e\xi\cdot A(x+\xi)\big) - - eC(\xi^2)\,\partial_\nu F^{\mu\nu}(x+\xi) - K_1(\xi^2)\, j^\mu(x+\xi) , \tag{3.30}$$

$\tilde C_8^{\mu\nu k}$ does not contribute any more because of (3.23), (3.24):

$$C\tilde c(\xi^2)\xi_\nu F^{\nu\mu}\, C^{-1} = -\tilde C(\xi^2)\xi_\nu F^{\nu\mu} = \tilde C(\xi^2)(-\xi_\nu)F^{\nu\mu} , \tag{3.31}$$

which contradicts (3.16) and therefore $\tilde C(\xi^2) = 0$. Now since

$$j^\mu(x) = \lim_{\xi\to 0} j^\mu(x+\xi_1-\xi) \tag{3.32}$$

one can construct a manifestly charge conjugation invariant form of $j_\mu(x)$:

$$\tilde j^\mu(x;\xi) \equiv \tfrac12[j^\mu(x;\xi) + j^\mu(x+\xi_1-\xi)] + \text{terms which} \to 0 \text{ as } \xi\to 0 , \tag{3.33}$$

$$\tilde j^\mu(x;\xi) = \frac{e}{2}[\bar\psi(x)\gamma^\mu\psi(x+\xi) - \gamma^\mu\psi(x)\bar\psi(x+\xi)] - e^2J^\mu(\xi)\,\xi\cdot A - - eC(\xi^2)\,\partial_\nu F^{\mu\nu} - K_1(\xi^2)\, j^\mu(x) , \tag{3.34}$$

where

$$j^\mu(x) \equiv \lim_{\xi\to 0}\tilde j^\mu(x;\xi) .$$

In the following we shall always use this current.

Applying gauge transformations on $\tilde j^\mu(x;\xi)$, we obtain a number of useful relations, which will be of importance in the following:

$$\tfrac12[\bar\psi(x)\gamma^\mu\psi(x+\xi) - \gamma^\mu\psi(x)\bar\psi(x+\xi)]\xi - eJ^\mu(\xi)(\xi\cdot A)\xi = 0 . \tag{3.35}$$

From

$$\lim_{\xi\to 0}[j^\mu(x;\xi) - j^\mu(x+\xi, -\xi)] = 0$$

one finds ([17])

$$
(3.36)\qquad \begin{cases} \frac{1}{2}\,[\bar{\psi}(x)\gamma^{\mu}\psi(x+\xi)+\gamma^{\mu}\psi(x)\bar{\psi}(x+\xi)]=ieJ^{\mu}(\xi)\,, \\ \frac{1}{2}\,[\bar{\psi}(x)\gamma^{\mu}\psi(x+\xi)+\gamma^{\mu}\psi(x)\bar{\psi}(x+\xi)]\xi\xi=0\,. \end{cases}
$$

These relations are needed for the derivation of the equal-time limit.

Now, only the determination of $C(\xi^2)$ and $K_1(\xi^2)$ remains. To specify these terms, we are going to use equal-time commutation relations, *i.e.* we resort to Lagrangian field theory. Then it is not difficult to obtain $C(\xi^2)$ and $K_1(\xi^2)$ in terms of Z_3 and Z_1, the usual charge and wave function renormalization constants, respectively. We have just to compare our renormalized current with the current that was given by KÄLLÉN ([26]) and which involves Z_1 and Z_2. It is then very easy to read off $C(\xi^2)$ and $K_1(\xi^2)$ in terms of these renormalization constants. Thus we find that the current is completely specified by these constants. Then we calculate $[j_0(x,t), j_1(y,t)]$ and transform back to the unrenormalized quantities. We will then see that the renormalization constants drop out, and we arrive at (1.1).

Let us list the commutation relations we will need in the following ([17,26,27]):

$$
(3.37)\qquad \{\psi_{\alpha}(x),\,\psi_{\beta}(x')\}_{t=t'}=\{\bar{\psi}_{\alpha}(x),\,\bar{\psi}_{\beta}(x')\}_{t=t'}=0\,,
$$

$$
(3.38)\qquad \{\bar{\psi}_{\alpha}(x),\,\psi_{\beta}(x')\}_{t=t'}=\gamma_0 Z_1^{-1}\delta(\boldsymbol{x}-\boldsymbol{x}')\,,
$$

$$
(3.39)\qquad [A_{\mu}(x),\,\psi_{\alpha}(x')]_{t=t'}=0\,,
$$

$$
(3.40)\qquad [A_{\mu}(x),\,A_{\nu}(x')]=0\,,
$$

$$
(3.41)\qquad [\partial_0 A_{\mu}(x),\,A_{\nu}(x')]_{t=t'}=-i[Z_3^{-1}g_{\mu\nu}-(Z_3^{-1}-1)g_{\mu 0}g_{\nu 0}]\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,,
$$

$$
(3.42)\qquad [\partial_0 A_{\mu}(x),\,\partial_0' A_{\nu}(x')]_{t=t'}=-(Z_3^{-1}-1)(g_{\mu 0}\partial_{\nu}+g_{\nu 0}\partial_{\mu})\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,.
$$

([26]) G. KÄLLÉN: *Helv. Phys. Acta*, 25, 417 (1952).

([27]) In view of all that one knows about the dimensional arguments, one might wonder whether the equal-time commutation relations are compatible with our dimensional assumptions. This is indeed the case because we have

$$
\bar{\psi}(x)\psi(y)\sim(x-y)^{-1}\,,\qquad A_{\mu}(x)A_{\nu}(y)\sim(x-y)^{0}\,,\qquad A_{\mu}(x)\psi(y)\sim(x-y)^{-\frac{1}{2}}\,,
$$
$$
\partial_0 A^{\mu}(x)A_{\nu}(y)\sim(x-y)^{-1}\,,\qquad \partial_0 A_{\mu}(x)\,\partial_0 A_{\nu}(y)\sim(x-y)^{-2}\,.
$$

Now, remembering that $\dim(\delta(\boldsymbol{x}-\boldsymbol{y}))=1$ and that leading singularities occur as mass independent coefficients of local field operators, the equal-time commutation relations by locality must have the form

$$
\sum_{n=0}^{N} E_n(x)\,\partial^n\,\delta(\boldsymbol{x}-\boldsymbol{y})\,.
$$

If we compare this with (3.37)-(3.42) we see that this is indeed the case and we also observe that no operator terms like $A_{\mu}(x)\,\delta(\boldsymbol{x}-\boldsymbol{y})$ can appear because of charge congation invariance. Thus (3.37)-(3.42) are all the commutation relations we can have.

Also we need the following spectral representation:

$$(3.43)\qquad G_{\alpha\beta}(x-y)=i\langle 0|\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}|0\rangle=\int_{-\infty}^{+\infty}\mathrm{d}\varkappa\,[\delta(\varkappa-m)+\sigma(\varkappa)]\,S_{\alpha\beta}(x-y,\varkappa)\,,$$

$$(3.44)\qquad Z_1^{-1}=1+\int\mathrm{d}\varkappa\,\sigma(\varkappa)\,,$$

$$(3.45)\qquad Z_3^{-1}=1+\int\frac{\mathrm{d}a\,\pi(a)}{a}\,.$$

KÄLLÉN [26] has given the following expression for the renormalized current:

$$(3.46)\qquad j_\mu(x)=\frac{e}{2}[\bar{\psi}(x),\gamma_\mu\psi(x)]+Z_1^{-1}(1-Z_3)\partial^\nu F_{\mu\nu}(x)+(1-Z_3^{-1})j_\mu(x)\,.$$

If we compare this with (3.34), then we see that, apart from the term $-e^2J_\mu(\xi)\cdot$ $\cdot(\xi\cdot A)$, which ensures current conservation—as we have seen, we get exactly the same expression. Therefore we identify

$$(3.47)\qquad \lim_{\xi\to 0}-eC(\xi^2)=Z_1^{-1}(0)\big(1-Z_3(0)\big)\,,$$

$$(3.48)\qquad \lim_{\xi\to 0}-K_1(\xi^2)=1-Z_1^{-1}\,,$$

and we find, setting $\xi_0=0$, *i.e.* using spacelike ξ only (because we do not want to spoil the canonical formalism):

$$(3.49)\qquad j_0(x;\xi)=\frac{e}{2}[\bar{\psi}(x)\gamma_0\psi(x+\xi)-\gamma^0\psi(x)\bar{\psi}(x+\xi)]+ \\ +Z_1^{-1}(1-Z_3)\partial^\nu F_{0\nu}+(1-Z_1^{-1})j_0(x)\,.$$

$J_0(\xi)$ vanishes, because, by covariance [17,18] $J_\mu(\xi)=J_\mu(\xi^2)\xi_\mu$

$$(3.50)\qquad j_1(x;\xi)=\frac{e}{2}[\bar{\psi}(x)\gamma_1\psi(x+\xi)-\gamma_1\psi(x)\bar{\psi}(x+\xi)]- \\ -e^2J_1(\xi)(\xi A)+Z_1^{-1}(1-Z_3)\partial^\nu F_{1\nu}(x)+(1-Z_1^{-1})j_1(x)\,.$$

To show that $K_1(\xi^2)$ is really given by (3.48), we use (3.38) and the fact that e is the physical charge corresponding to the validity of the relation

$$(3.51)\qquad [\psi(\boldsymbol{x},t),\,j_0(\boldsymbol{y},t)]=e\psi(\boldsymbol{x},t)\,\delta(\boldsymbol{x}-\boldsymbol{y})\,.$$

Defining the charge operator by

$$Q = \int \mathrm{d}y\, j_0(y, t) \tag{3.52}$$

and integrating (3.51) over y leads to

$$[\psi(x, t), Q] = e\psi(x, t)\,, \tag{3.53}$$

as we wanted to show.

Now using (3.49) and

$$[\psi(x), \partial_0 A_\mu(x')]_{t=t'} = 0\,, \tag{3.54}$$

which one easily derives (17) and using also

$$\partial^\nu F_{0\nu} = -\partial^1\partial^0 A^1 + \partial^{12} A^0\,,$$

one obtains

$$\lim_{\xi\to 0}\,[\psi(x), j^0(y, \xi)]_{x_0=y_0} = eZ_1^{-1}\psi(x)\delta(x-y) - eK_1(0)\psi(x)\delta(x-y)\,.$$

Now, requiring again that (3.51) gives (3.48) as a check, we are in a position to calculate $[j_1(x, t), j_0(y, t)]$. Using

$$\partial^\nu F_{1\nu} = -\partial^0\partial^1 A^0 + \partial^{02} A^1 = -\partial^0\partial^1 A^0 + j^1 + \partial_0^2 A^1 \tag{3.55}$$

and

$$(1-Z_3)^2 Z_3^{-1}[(-\partial^1\partial^0 A^1(x) + \partial^{12} A^0(x)), (\partial^1\partial^0 A^0(y) - \partial^{12} A^1(y))]_{x_0=y_0} = 0\,, \tag{3.56}$$

after bringing $j^1(x)$, of eq. (3.55), over to the other side of (3.50). For (3.56) we needed eqs. (3.41) and (3.42).

As a next step we calculate

$$\begin{aligned} e^2 Z_1 Z_3^{-1} J^1(\xi)\xi^1[A^1(x), (1-Z_3)(-\partial_y^1)\partial^0 A^1(y)]_{x_0=y_0} &= \\ &= -ie^2(1-Z_3)Z_3^{-2} J^1(\xi)\xi^1\partial_x^1\delta(x-y) \end{aligned} \tag{3.57}$$

using (3.41).

Let us take into account now the following commutators (17,27)

$$[\partial_0\psi(x), A_0(x')]_{t=t'} = 0\,, \tag{3.58}$$

$$[j_1(x), A_{1,0}(x')]_{t=t'} = 0\,, \tag{3.59}$$

$$[j_1(x), A_{0,0}(x')]_{t=t'} = 0\,, \tag{3.60}$$

$$[j_0(x), A_{\mu,\nu}(x')]_{t=t'} = 0\,. \tag{3.61}$$

We finally get

$$(3.62)\qquad \lim_{\xi'\to 0}[j_1(x;\xi), j_0(x',\xi')]_{t=t'} = -\frac{e^2}{2} Z_1 Z_3^{-1}[\bar\psi(x)\gamma_1\psi(x+\xi)+\psi(x)\gamma_1\bar\psi(x+\xi)]\cdot \cdot\xi_1\partial_1\delta(x-x') - ie^2 Z_1(1-Z_3)Z_3^{-2}J_1\xi_1\partial_1\delta(x-x')\,.$$

Using now (3.36), we find

$$(3.63)\qquad \lim_{\xi'\to 0}[j_1(x,\xi), j_0(x',\xi')]_{t=t'} = -ie^2 Z_1 Z_3^{-1}\xi_1\partial_1 J_1(\xi)\delta(x-x') - -ie^2 Z_1(1-Z_3)Z_3^{-2}J_1\xi_1\partial_1\delta(x-x') = -ie^2 Z_1 Z_3^{-2} J_1(\xi)\xi_1\partial_1\delta(x-x')\,.$$

We have made here the mild assumption that we can calculate the various commutation relations by interchanging the $\xi\to 0$ limit with the equal-time limit, which is justified if we compare with BRANDT ([17]). Let us now transform back to the unrenormalized quantities. We have

$$(3.64)\qquad \begin{cases} A_\mu = A_\mu^{\text{un}} Z_3^{-\frac12}\,, & j_\mu^{\text{un}} = Z_3^{\frac12} j_\mu\,,\\ e\ = Z_3^{\frac12} e_0\,, & j_\mu = Z_3^{-\frac12} j_\mu^{\text{un}}\,.\\ \psi\ = Z_1^{-\frac12}\psi^{\text{un}}\,, & \end{cases}$$

Remembering that

$$(3.65)\quad \xi_1 J_1(\xi) = \int d\varkappa\big(\delta(\varkappa - m)+\sigma(\varkappa)\big)\,\mathrm{tr}\gamma_1 S^{\text{free}}(\xi)\xi_1 = -Z_1^{-1}\,\mathrm{tr}\,\gamma_1 S^{\text{free}}(\xi)\xi_1 = -\frac{Z_1^{-1}}{\pi}\,,$$

where we used ([9])

$$S^{\text{free}}(\xi) = -\frac{\gamma\cdot\xi}{2\pi\xi^2} = -\frac{\gamma_1\xi_1}{\xi_1}\,,$$

we get

$$Z_3^{-1}[j_1^{\text{un}}(x), j_0^{\text{un}}(y)]_{x_0=y_0} = +ie_0^2 Z_3 Z_1 Z_3^{-2} Z_1^{-1}\partial_1\delta(x-y)\,,$$

and therefore ([28])

$$(3.66)\qquad [j_1^{\text{un}}(x), j_0^{\text{un}}(y)]_{x_0=y_0} = +\frac{i}{\pi}e_0^2\partial_1\delta(x-y)\,.$$

([28]) Dr. D. GROSS has pointed out to me that since from $\partial_\mu j^\mu(x)=0$ and $\partial_\mu j_5^\mu(x) = -2mi\bar\psi\gamma_5\psi$, $j_5^\mu(x)=\partial^\mu\sigma(x)$ follows ([2]), and since $j^0=-j_5^1=\partial^1\sigma$, $j^1=-j_5^0=\partial^0\sigma$:

$$[j^{05}(x), j^{15}(x')]_{t=t'} = \partial'^1[\partial_0\sigma(x),\sigma(x')]_{t=t'} = +\frac{i}{\pi}\partial^1\delta(x-x')\,,$$

so that $\sigma(x)$ has to be a canonical field. This is here only true for the *free field* case (ε), because we used the free field equations. In a subsequent paper it will be shown that it holds for the interacting case also.

So we see that we have obtained just the same result as for the case of vanishing fermion mass (5) or for the case of lowest-order perturbation calculation (9). That this result must be true is shown by the following simple argument. Since $j_\mu(x)$ is gauge invariant, as we have shown, the commutator $[j_0(x), j_1(y)]_{x_0=y_0}$ has to be also gauge invariant. This implies that it cannot depend on Z_1, which is a gauge-variant quantity and therefore has to cancel out, as we have seen. Our commutator could depend on Z_3. But this is also not possible as Z_3 vanishes for m going to zero (remember that in the Schwinger model (5) no photons exist, we have instead a massive stable vector boson). Since we know that $[j_0(x), j_1(y)]_{x_0=y_0}$ exists for $m\to 0$ (this is just the exact result (5) in this case), we immediately see that only the result (3.66) can arise.

The surprising result we obtained was that there are no operator Schwinger terms and no further terms, even for $m\neq 0$. All the amplitudes which start at higher order as the three-photon vertex, etc., have no Schwinger term; they are gauge invariant without introducing counter terms. This suggests that the solution of this model must be quite simple and one should be able to solve it exactly.

From our basic result (3.66), we can now derive an interesting sum rule. We know that the vacuum polarization tensor has the following spectral representation (29):

$$\Pi^{\text{un}}_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\int \frac{\mathrm{d}a^2\,\varrho^{\text{un}}(a^2)}{a^2 - q^2}\,. \tag{3.67}$$

Since, on the other hand,

$$\Pi^{\text{un}}_{\mu\nu}(x-x') = \langle 0|Tj^{\text{un}}_\mu(x)\,j^{\text{un}}_\nu(x')|0\rangle = \frac{1}{(2\pi)^2}\int \exp[i-iq(x-x')]\,\Pi^{\text{un}}_{\mu\nu}(q)\,. \tag{3.68}$$

We can use the Johnson-Low-Bjorken (30) argument to relate the spectral integral for $\Pi^{\text{un}}_{\mu\nu}(q)$ to the equal-time commutator. We find

$$\text{Schwinger term} = \int_0^\infty \mathrm{d}a^2\,\varrho^{\text{un}}(a^2)(g_{\mu 0}g_{\nu 0} - g_{\mu\nu})\,. \tag{3.69}$$

Now using (3.66) and (3.68), we get

$$\int_0^\infty \mathrm{d}a^2\,\varrho^{\text{un}}(a^2) = \frac{e_0^2}{\pi}\,. \tag{3.70}$$

---

(29) See: G. KÄLLÉN: *Handbuch der Physik*, vol. 5/1 (1958), for example.

(30) K. JOHNSON: *Nucl. Phys.*, 25, 431 (1961); K. JOHNSON and F. LOW: *Suppl. Progr. Theor. Phys.*, 37, 38, 74 (1966); J. BJORKEN: *Phys. Rev.*, 148, 1467 (1966).

This is our main result. It is an exact sum rule. Let us split

$$\varrho^{\text{un}}(a^2) = \varrho^{(0)\text{un}}(a^2) + \varrho^{\text{un rest}}(a^2) \tag{3.71}$$

and using ([9])

$$\varrho^{(0)\text{un}}(a^2) = 2a^2 m_0^2 \Big/ \left(\pi a^4 \sqrt{1 - \frac{4m_0^2}{a^2}}\right),$$

we find

$$\int_0^\infty \varrho^{\text{un rest}}(a^2)\, \mathrm{d}a^2 = 0\,. \tag{3.72}$$

From ref. ([9]), one can also convince oneself that the fourth-order correction to the Schwinger term does indeed vanish. The same is true for the sixth-order contribution ([20]). Thus the result (3.72) is verified in perturbation theory through sixth order. Unfortunately, although (3.17) is an exact result, we cannot conclude (*) that $\varrho^{\text{un rest}}(a^2)$ is identically zero, as we would then have an exact result for the vacuum polarization tensor. It would be enough to know that $\varrho^{\text{un}(0)}(a^2) > \varrho^{\text{un rest}}(a^2)$, but up to now we have not succeeded in showing this. Nevertheless (3.72) suggests that $\Pi^{\text{un}}_{\mu\nu}(q)$ must have a very simple form.

One can also try to use dimensional arguments and the exact result for $m_0 = 0$, to get more information on $\varrho^{\text{un rest}}(a^2)$, but it is very easy (**) to find counterexamples. We therefore have to solve the basic equations, especially the fermion Green function in an external electromagnetic field ([31]) to get further information.

4. – Conclusions.

We have proved, following the interesting work of BRANDT, that gauge invariance, equal-time commutation relations and charge-conjugation invariance, specify the current in electrodynamics completely in two-dimensional space time. Using this, we were able to derive an exact result for the Schwinger term, which gave rise to an exact sum rule for the vacuum polarization tensor. This sum rule gives a strong indication that this model can be solved explicitly and that it is therefore worth-while to study it. Since it has probably

(*) I thank Prof. S. COLEMAN for stressing this point to me.

(**) Dr. R. JACKIW provided these counterexamples and I thank him very much for his critical interest.

([31]) P. L. F. HARBERLER: to be published.

a nontrivial S-matrix (2) and interesting (32) physical properties, we believe that the exact solution of this model also gives us a clue for solving the dynamical problem in four-dimensional space time.

* * *

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APPENDIX A

Proof that the Ward identity is already valid in lowest order.

$$\Gamma_\mu(p,0)=e_0\partial_\mu G^{-1}(p) \qquad \text{or} \qquad \Lambda_\mu(p,p)=e_0\partial^\mu\Sigma(p)\,, \tag{A.1}$$

$$\Sigma(p)=\frac{m_0e_0^2}{\pi}\int\limits_{(m_0+\mu_0)^2}^{\infty}\frac{\mathrm{d}a^2}{\sqrt{(a^2-m_0^2-\mu_0^2)^2-4m_0^2\mu_0^2}\,(a^2-p^2)}\,, \tag{A.2}$$

$$\left\{\begin{aligned}&\Lambda^\mu(p,p)=-\frac{m_0e_0^3}{\pi}p^\mu\left\{\frac{p^2-m_0^2-\mu_0^2}{[(p^2-m_0^2-\mu_0^2)^2-4m_0^2\mu_0^2]^{\frac{3}{2}}}\log\frac{p^2-m_0^2-\mu_0^2-\sqrt{\cdot/.}}{p^2-m_0^2-\mu_0^2+\sqrt{\cdot/.}}+\right.\\&\qquad\qquad\qquad\qquad\qquad\qquad\left.+\frac{2}{(p^2-m_0^2-\mu_0^2)^2-4m_0^2\mu_0^2}\right\},\\&\sqrt{\cdot/.}=[(p^2-m_0^2-\mu_0^2)^2-4m_0^2\mu_0^2]^{\frac{1}{2}}\,.\end{aligned}\right. \tag{A.3}$$

It is easy to see that $e_0(\partial/\partial p_\mu)\Sigma(p)$ is just given by (A.3). (m_0, μ_0 and e_0 are the unrenormalized quantities.)

It therefore follows that $D_2^\mu(\eta)$ and $D_3(\eta)$ are actually not needed because the Ward identity is already fulfilled.

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(32) P. L. F. HARBERLER and I. SAAVEDRA: *Nuovo Cimento*, 49 A, 194 (1967); P. L. F. HABERLER: *Nuovo Cimento*, 47 A, 929 (1967).

Appendix B

The photon amplitude is given by

$$\Pi_{\mu\nu\lambda\sigma}(k_1k_2k_3k_4) = 2T_{\mu\nu\lambda\sigma}(k_1k_2k_3k_4) + 2T_{\mu\nu\sigma\lambda}(k_1k_2k_4k_3) + 2T_{\mu\lambda\nu\sigma}(k_1k_3k_2k_4)\,, \tag{B.1}$$

where, to lowest order, one finds

$$T_{\mu\nu\lambda\sigma} = -\frac{e_0^4}{(2\pi)^2}\int \mathrm{d}^2p\, \mathrm{tr}\,\gamma^\mu \frac{1}{\gamma\cdot p - m_0}\gamma^\nu \frac{1}{\gamma\cdot p - (\gamma\cdot k)_2 - m_0}\cdot \\ \cdot\gamma^\lambda \frac{1}{\gamma\cdot p - (\gamma\cdot k)_2 - (\gamma\cdot k)_3 - m_0}\gamma^\sigma \frac{1}{\gamma\cdot p + (\gamma\cdot k)_1 - m_0}\,, \tag{B.2}$$

with $k_1 + k_2 + k_3 + k_4 = 0$.

The first thing we want to show is that $\Pi_{\mu\nu\lambda\sigma}(0,0,0,0) = 0$. We find

$$\Pi_{\mu\nu\lambda\sigma}(0,0,0,0) = \frac{e_0^4 m_0^2}{(2\pi)^2}\,\mathrm{d}^2p\,[\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda + \gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma + \gamma^\mu\gamma^\lambda\gamma^\nu\gamma^\sigma]\cdot \\ \cdot\left\{\frac{2}{(p^2 - m_0^2)^3} + \frac{3m_0^2}{(p^2 - m_0^2)^4}\right\}.$$

Performing the integration gives exactly zero.

In the same way, using the Ward identity, (A.1), one shows that

$$\Pi_{\mu\nu\lambda\sigma}(0, k_2, k_3, k_4) = \ldots = \Pi_{\mu\nu\lambda\sigma}(k_1, k_2, k_3, 0) = 0$$

and also that of course

$$k_1^\mu \Pi_{\mu\nu\lambda\sigma}(k_1k_2k_3k_4) = \ldots = k_4^\sigma \Pi_{\mu\nu\lambda\sigma}(k_1k_2k_3k_4) = 0\,.$$

Here one uses

$$\frac{1}{\gamma\cdot p - m_0}\gamma\cdot k\frac{1}{\gamma\cdot p - \gamma\cdot k - m_0} = -\frac{1}{\gamma\cdot p - m_0} + \frac{1}{\gamma\cdot p - \gamma\cdot k - m_0} \tag{B.3}$$

and then redefining the integration variable, one is allowed to shift the integration variable because all integrals are convergent in two-dimensional space-time, and one finds that the terms cancel pairwise. We also found the somewhat surprising result that $\Pi_{\mu\nu\lambda\sigma}(k_1k_2k_3k_4) = 0$ for $k_1 = k_2$ and $k_3 = k_4$, *i.e.* if it depends only on one vector k_μ, then $\Pi_{\mu\nu\lambda\sigma}(k)$ vanishes identically. For $\Pi_{\mu\nu\lambda\sigma}(k_1k_2k_3k_4)$ to vanish as a whole, $[j_\mu^{\text{free}}(x), j_\nu^{\text{free}}] \doteq c$ number, which is not the case ([33]) here. In any case $\Pi_{\mu\nu\lambda\sigma}(k_1k_2k_3k_4)$ must have a very simple form also and it does not contribute to the Schwinger term.

([33]) This is due to a theorem of Borchers and Robinson; V. Glaser and H. Epstein: private communication.

Appendix C

In this last Appendix, we want to calculate $\langle 0|j_\mu(x)|0\rangle$ up to second order in e_0. We have [34]

$$\text{(C.1)} \qquad \langle 0|j_\mu(x)|0\rangle = -e_0 \operatorname{tr} i\gamma_\mu S(x,x,A) = -e_0 \cdot$$

$$\cdot \left\{ \operatorname{tr} i\gamma \lim_{x\to x'} \left[S_0(x-x_0') + e_0 \int \mathrm{d}y'\, S_0(x-y')\gamma_\nu S(y',x',A) A^\nu(y') \right] \exp\left[ie_0 \int_x^{x+\varepsilon} \mathrm{d}\xi_\nu A^\nu \right] \right\}.$$

In lowest order we have

$$\text{(C.2)} \qquad \langle 0|j_\mu(x)|0\rangle = -ie_0 \operatorname{tr}\gamma_\mu \lim_{x\to x'} \cdot$$

$$\cdot \left[S_0(x-x') + e_0 \int \mathrm{d}y'\, S_0(x-y')\gamma_\nu S_0(y'-x) A^\nu(y') \right] \cdot \exp\left[ie_0 \int_x^{x+\varepsilon} \mathrm{d}\xi_\nu A^\nu \right];$$

we always have ε spacelike, and take the symmetrical limit. If we develop in small ε, we get

$$\langle 0|j_\mu(x)|0\rangle = -ie_0 \operatorname{tr}\gamma_\mu \lim_{x\to x'} \cdot$$

$$\cdot \left[S_0(x-x') + e_0 \int \mathrm{d}y'\, S_0(x-y')\gamma_\nu S_0(y'-x) A^\nu(y') + ie_0 S_0(x-x') \int_x^{x+\varepsilon} \mathrm{d}\xi_\nu A^\nu \right].$$

The first vanishes by symmetry, because $S_0(\varepsilon) \sim 1/\varepsilon$. Therefore we also need only the first term of the line integral and thus we get

$$\langle 0|j_\mu(x)|0\rangle = -ie_0 \operatorname{tr}\gamma_\mu \left[e_0 \int \mathrm{d}y'\, S_0(x-y')\gamma_\nu S_0(y'-x') A^\nu(y') + ie_0 S_0(\varepsilon)\,\varepsilon\cdot A \right],$$

where [9]

$$S_0(\varepsilon) = -\frac{\gamma\cdot\varepsilon}{2\pi\varepsilon^2}, \qquad \operatorname{tr}\gamma_\mu\gamma\cdot\varepsilon = 2\varepsilon_\mu,$$

$$ie_0 \operatorname{tr}(-ie_0)\frac{\gamma\cdot\varepsilon\varepsilon\cdot A}{-2\pi\varepsilon^2} = -\frac{e_0^2}{\pi}\frac{\varepsilon_\mu\varepsilon\cdot A}{\varepsilon^2} = -\frac{e_0^2}{\pi}[\delta_{\mu 0}\varepsilon_0 + \delta_{\mu 1}\varepsilon_1]\frac{\varepsilon_1 A_1}{\varepsilon_1^2} = -\frac{e_0^2}{\pi}\delta_1 A_1,$$

$$ie_0 \operatorname{tr}(-ie_0)\frac{\gamma\varepsilon\varepsilon A}{-2\pi\varepsilon^2} = -\frac{\pi}{e_0^2}(g_{\mu\nu} - \delta_{\mu 0}\delta_{\nu 0})A^\nu(x) =$$

$$= -\frac{e_0^2}{\pi}(g_{\mu\nu} - \delta_{\mu 0}\delta_{\nu 0}) \int \mathrm{d}y'\, \delta(x-y') A^\nu(y').$$

([34]) K. Johnson: *Lectures on Particles and Field Theory*, vol. 2, *Brandeis Lectures, 1964*, edited by S. Deser and K. Ford (Englewood Cliffs., N. J., 1965).

Now we find (⁹)

$$\langle 0|j_\mu(x)|0\rangle = \frac{4e_0^2 m_0^2}{\pi}\cdot$$

$$\cdot\left[\int_{2m_0}^{\infty}\frac{ds}{s^2\sqrt{s^2-4m_0^2}}\int dy\{(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\Delta_F(x-y', s) - \delta(x-y')(\delta_{\mu 0}\delta_{\nu 0}-g_{\mu\nu})\}A^\nu(y')\right] -$$

$$-\frac{e_0^2}{\pi}(g_{\mu\nu}-\delta_{\mu 0}\delta_{\nu 0})\int dy'\,\delta(x-y')A^\nu(y') =$$

$$= \frac{4e_0^2 m_0^2}{\pi}\left[\int_{2m_0}^{\infty}\frac{ds}{s^2\sqrt{s^2-4m_0^2}}\int dy'(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\Delta_F(x-y', s)A^\nu(y')\right],$$

which is the desired result. The reason for deriving this in such detail is partly because one finds great confusion on this point in the literature. We also wanted to check that only spacelike ε are needed.

RIASSUNTO (*)

Si dimostra, usando i risultati di Brandt, che nell'elettrodinamica spinoriale bidimensionale, con la massa del fermione che non si annulla, $[j_0^{un}(x, t), j_1^{un}(y, t)] = -(i/\pi)e_0^2\,\partial_1\delta(x-y)$, dove $j_\mu^{un}(k)$ è l'esatta corrente elettromagnetica non rinormalizzata. Si discutono le implicazioni per la soluzione esatta della polarizzazione del vuoto.

(*) *Traduzione a cura della Redazione.*

Точный результат в двумерной квантовой электродинамике с ненулевой массой фермиона.

Резюме (*). — Используя результаты Брандта, в двумерной спинорной электродинамике с не обращающейся в нуль массой фермиона доказывается, что $[j_0^{un}(x, t), j_1^{un}(y, t)] = +(i/\pi)e_0^2\partial_1\delta(x-y)$, где $j_\mu^{un}(k)$ представляет точный неперенормированный электромагнитный ток. Обсуждаются применения для точного решения поляризации вакуума.

(*) *Переведено редакцией.*